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Quantization in Curvilinear Coordinates

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Two prescriptions often used to find Hermitian operators corresponding to classical quantities can be removed. Components of momentum are of three types, linear momentum $P^k_{\langle q\rangle}$, canonical momentum $P_{(q)k}$, and generalized momentum P^k_q. Using metrical geometry, their mutual relations are established. The operators $\hat{P}^k_{\langle q\rangle}$ and $\hat{P}_{(q)k}$ are given by substituting quantum commutation brackets for classical Poisson brackets. The relations among classical quantities are divided into two types according to whether they have physical meaning. Those which have physical meaning go over into the corresponding operator relations.

1. INTRODUCTION

If $G(q, p_{(q)})$ is a scalar function of canonical variables $q(q^1, q^2, q^3)$ and $P_{(q)}(P_{(q)1}, P_{(q)2}, P_{(q)3})$, it is well known that $G(q, p_{(q)})$ obeys the classical dynamical equation

$$\dot{G}(q, p_{(q)}) = \{G(q, p_{(q)}), H(q, p_{(q)})\} \tag{1}$$

where $H(q, p_{(q)})$ is the Hamiltonian of the classical system, and

$$\{G(q, p_{(q)}), H(q, p_{(q)})\} = \left(\frac{\partial G}{\partial q^i}\frac{\partial H}{\partial p_{(q)i}} - \frac{\partial H}{\partial q^i}\frac{\partial G}{\partial p_{(q)i}}\right)$$

is the Poisson bracket (abbreviated as PB) for $G(q, p_{(q)})$ and $H(q, p_{(q)})$. Further, the quantum equation of motion corresponding to a given classical system can be found by substituting the commutator bracket divided by $i\hbar$ for the PB,

$$\dot{G} = \{G, H\} \to \hat{\dot{G}} = \frac{1}{i\hbar}[\hat{G}, \hat{H}] \tag{2a}$$

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$$P_{(q)k} \rightarrow \hat{P}_{(q)k} = \frac{\hbar}{i} \frac{\partial}{\partial q^k} \tag{2b}$$

where $\hat{P}_{(q)k}$, $\hat{G}$, and $\hat{H}$ are operators corresponding to the classical quantities $P_{(q)k}$, $G(q, p_{(q)})$, and $H(q, p_{(q)})$, respectively, and

$$[\hat{G}, \hat{H}] = (\hat{G}\hat{H} - \hat{H}\hat{G})$$

However, it was found that the quantization program of equations (2a) and (2b) is not always valid in an arbitrarily chosen coordinate system. Hence, Schiff (1968) pointed out the following two prescriptions:

"First, the coordinates and momenta must be expressed in Cartesian coordinates. Second, ambiguities in the order of noncommuting factors are usually resolved by taking a symmetric average of the various possible orders."

These two prescriptions will simply be called the "(x)-condition" and the "symmetrization procedure," respectively.

Indeed, it can be seen why the (x)-condition seems to be a reasonable restriction when we carry out the substitution from classical quantities to quantum operators in spherical polar coordinates. Suppose that the Cartesian coordinates (x) and the rectangular coordinates (q) are static relative to each other without loss of generality, let us consider a single particle moving in a conservative potential energy field $U(x)$. Evidently, $U(x) = U(q)$. Then the Lagrangian $L(x, \dot{x})$ of the single particle is

$$L(x, \dot{x}) = \frac{\mu}{2}[(\dot{x}^1)^1 + (\dot{x}^2)^2 + (\dot{x}^3)^2] - U(x) \tag{3a}$$

$$L(q, \dot{q}) = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta\dot{\varphi}^2) - U(q) \tag{3b}$$

and $L(x, \dot{x}) = L(q, \dot{q})$. In accordance with the definitions of the canonical momentum and Hamiltonian, we have

$$P_{(x)i} = \frac{\partial L}{\partial \dot{x}^i} \qquad (i = 1, 2, 3) \tag{4a}$$

$$P_{(s)r} = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}, \qquad P_{(s)\theta} = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$$

$$P_{(s)\varphi} = \frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \sin^2 \theta\dot{\varphi} \tag{4b}$$

and

$$H(x, p_{(x)}) = P_{(x)i}\dot{x}^i - L(x, \dot{x})$$

$$= \frac{1}{2\mu}(P^2_{(x)1} + P^2_{(x)2} + P^2_{(x)3}) + U(x) \quad (5a)$$

$$H(q, p_{(q)}) = P_{(s)k}\dot{q}^k - L(q, \dot{q})$$

$$= \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2) + U(q) \quad (5b)$$

and $H(x, p_{(x)}) = H(q, p_{(q)})$. Obviously, equation (5b) can be expressed in terms of canonical momenta $P_{(s)k} = \partial L/\partial \dot{q}^k$ and generalized momenta $P_s^k = \mu\dot{q}^k$, respectively. We have

$$H(q, p_{(s)}) = \frac{1}{2\mu}\left(P^2_{(s)r} + \frac{1}{r^2}P^2_{(s)\theta} + \frac{1}{r^2\sin^2\theta}P_{(s)\varphi}\right) + U(r, \theta, \varphi) \quad (6a)$$

$$H(q, p_s) = \frac{1}{2\mu}(P^2_{sr} + r^2 P^2_{s\theta} + r^2\sin^2\theta\, P^2_{s\varphi}) + U(r, \theta, \varphi) \quad (6b)$$

Now, with the help of the quantization procedure of equation (2b), the Hamiltonian functions of equations (6a) and (6b) will be replaced by corresponding operators, respectively.

In the (x)-system, the canonical momentum operators of equation (2b) are

$$\hat{P}_{(x)i} = \frac{\hbar}{i}\frac{\partial}{\partial x^i}, \qquad i = 1, 2, 3$$

Substituting $\hat{P}_{(x)i}$ into equation (5a) gives

$$\hat{H}(\hat{x}, \hat{p}_{(x)}) = -\frac{\hbar}{2\mu}\left(\frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2}\right) + U(x) \quad (7)$$

In the (s)-system, i.e., spherical coordinates, if the operator of equation (2b) denotes operators of canonical momenta, i.e.,

$$\hat{P}_{(s)k} = \frac{\hbar}{i}\frac{\partial}{\partial q^x}, \qquad q = r, \text{ or } \theta, \text{ or } \varphi$$

then substituting $\hat{P}_{(s)k}$ into equation (6a) gives

$$\hat{H}(\hat{q}, \hat{p}_{(s)}) = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + U(r, \theta, \phi) \quad (8a)$$

If the operator of equation (2b) denotes operators of generalized momentum, i.e.,

$$\hat{P}_s^k = \frac{\hbar}{i}\frac{\partial}{\partial q^k}, \qquad q = r, \text{ or } \theta, \text{ or } \varphi$$

then substituting $\hat{P}_s^k$ into equation (6b) gives

$$\hat{H}(\hat{q}, \hat{p}_s) = -\frac{\hbar}{2\mu}\left(\frac{\partial^2}{\partial r^2} + r^2\frac{\partial^2}{\partial \theta^2} + r^2\sin^2\theta\frac{\partial^2}{\partial \varphi^2}\right) + U(r, \theta, \varphi) \tag{8b}$$

On the other hand, it is well known that the correct Hamiltonian operator in the (s)-system can be derived from $\hat{H}(\hat{x}, \hat{p}_{(x)})$ given in the (x)-system by the use of a compound differential, that is,

$$\hat{H}(\hat{q}, \hat{p}_{(q)}) = -\frac{\hbar}{2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] + U(r, \theta, \varphi) \tag{9}$$

We see that neither equation (8a) nor (8b) is the correct Hamiltonian operator in a spherical coordinate system. It seems as if the coordinates and momenta are only expressible in Cartesian coordinates. The transition procedures given in equations (2a) and (2b) can be considered reliable.

When the transformation laws of components of canonical momenta under transformation of the coordinate system are taken over to the corresponding operators, it seems as if the symmetrization procedure also is reasonable. It will be shown later that the transformation law of the canonical momenta is

$$P_{(x)i} = \frac{\partial q^k}{\partial x^i} P_{(q)k} = P_{(q)k}\frac{\partial q^k}{\partial x^i} \tag{10a}$$

The order of classical quantities $P_{(q)k}$ and $\partial q^k/\partial x^i$ is commutable, but the order of the operators $\hat{P}_{(q)k}$ and $\partial q^k/\partial x^i$ is not. For $P_{(x)i}$ on the left side of equation (10a) to go over into a Hermitian operator $\hat{P}_{(x)i}$, the symmetrization procedure needs to be applied to the right side of (10a) before the classical quantity is replaced by the corresponding quantum operator; that is,

$$P_{(x)i} = \frac{1}{2}\left(\frac{\partial q^k}{\partial x^i} P_{(q)k} + P_{(q)k}\frac{\partial q^k}{\partial x^i}\right) \tag{10b}$$

Only then can we carry out the transition from classical quantities to quantum operators. For clarity, let both sides of the operator equality corresponding to (10b) act on an arbitrary ket vector. We thus obtain

$$\hat{P}_{(x)i}|\rangle = \frac{1}{2}\left(\frac{\partial q^k}{\partial x^i}\hat{P}_{(q)k} + \hat{P}_{(q)k}\frac{\partial q^k}{\partial x^i}\right)|\rangle$$

The operators in the parentheses have been composed into a Hermitian operator. Thus, it seems as if the symmetrization procedure originating in the Hermitian requirement is also reasonable. After performing the operation, since the ket vector is arbitrary, we have

$$\hat{P}_{(x)i} = \left(\frac{\partial q^k}{\partial x^i}\hat{P}_{(q)k} + \frac{\hbar}{2i}\frac{\partial}{\partial q^k}\left(\frac{\partial q^k}{\partial x^i}\right)\right) \tag{11}$$

On the other hand, the correct transformation law for momentum operators in the (q)-system can be derived by the use of the rule of the compound differential,

$$\hat{P}_{(x)i} = \frac{\hbar}{i}\frac{\partial}{\partial x^i} = \frac{\hbar}{i}\frac{\partial q^k}{\partial x^i}\frac{\partial}{\partial q^k} = \frac{\partial q^k}{\partial x^i}\hat{P}_{(q)k} \tag{12}$$

The right side of equation (11) gives rise to some additional terms compared to equation (12). Evidently, the transformation expression between the coordinates of the (q)-system and the (x)-system is $q^k = (a_l^k x^l + b^k)$, and a_l^k, b^k are constant, i.e., the (q)-system must also be a Cartesian coordinate system or the additional terms will not vanish. It seems as if the symmetrization procedure must be performed in a Cartesian coordinate system.

The above arguments can be illustrated by the process in Figure 1. If we perform steps A and D in order, then we arrive at the correct operator $\hat{Q}'(\hat{q}, \hat{P}_{(q)})$ in the (q)-system, for example, the right-hand sides of equations (9) and (12). If the steps C and B are performed successively, though we

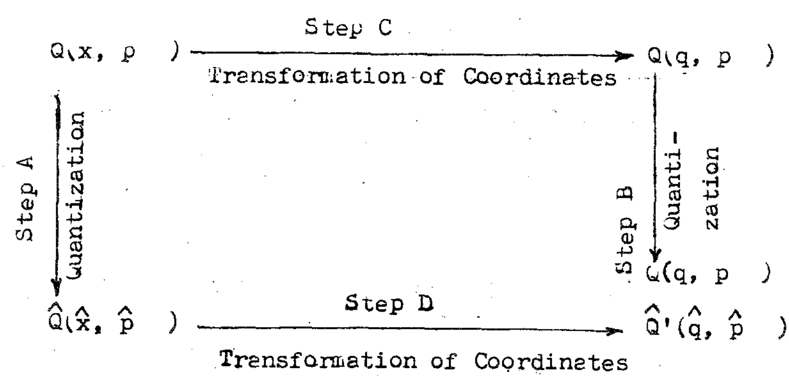

Fig. 1

may produce an operator $\hat{Q}(\hat{q}, \hat{p}_{(q)})$ in the (q)-system [such as the right-hand sides of (8a), (8b), and (11)], this operator $\hat{Q}(\hat{q}, \hat{p}_{(q)})$ is not always equal to the correct operator $\hat{Q}'(q, p_{(q)})$.

In order to seek a general quantization procedure without the restrictions of the (x)-condition and the symmetrization procedure, the coordinate systems have been examined in different ways, such as the method of covariant derivative or the method of infinitesimal contact transformation with classical analogy. In this way Merzbacher (1970) came to the following conclusion: the types of coordinate systems in which the canonical quantization program is valid are those which can be obtained from a Cartesian coordinate system by continuous succession of infinitesimal contact transformations (it may be proved that the quantization program in the PB formalism is similarly valid). Because the classical analogy might be not accurate, we tried to solve the same problem with the help of the infinitesimal contact transformations in a first-order approximation (unpublished work) and found that a coordinate condition under which equations (2a) and (2b) are valid is

$$[\hat{q}, \dot{\hat{G}}] = 0$$

(characterizing so-called "G-type coordinates") where $\hat{G}$ is a Hermitian operator corresponding to the generatrix function $G(q, p_{(q)})$ of infinitesimal contact transformations. We were only able to replace the (x)-condition by the G-type coordinates, while the symmetrization procedure remained unchanged.

The following discussion shows that, provided we apply exactly the metric analytical method to the transition from classical to quantum systems, not only can both the (x)-condition and the symmetrization procedure be entirely removed, but also only equation (2a) is necessary.

2. METRIC RELATION OF THREE TYPES OF CLASSICAL MOMENTA

Suppose that the two arbitrary systems of orthogonal coordinates (q^k) and (q'^r) $(k, r = 1, 2, 3)$ are static relative to each other and that there exist single-valued invertible transformation relations between them,

$$q^k = q^k(q'^1, q'^2, q'^3) = (q'), \qquad k = 1, 2, 3 \tag{13a}$$

$$q'^r = q'^r(q^1, q^2, q^3) = q'(q), \qquad r = 1, 2, 3 \tag{13b}$$

where the functions $q^k(q')$ and $q'^r(q)$ are assumed to possess continuous partial derivatives to the needed order. In the orthogonal coordinate systems

(q) and (q'), the metric components g_{kl} and g'_{rs} satisfy the conditions

$$\begin{aligned} g_{kl} &= 0 \qquad \text{for} \quad k \neq l \\ g'_{rs} &= 0 \qquad \text{for} \quad r \neq s \end{aligned} \tag{14}$$

Their contravariant components for the given point q in space are defined as the inverses of g_{ij} and g'_{rs}, i.e.,

$$g_{ij} g^{jl} = \delta_i^l \qquad \text{and} \qquad g'_{rs} g'^{st} = \delta_r^t \tag{15}$$

Using the metric components g_{kl} and g'_{rs}, the square of linear elements in 3-dimensional space ds^2 can be written as

$$ds^2_{\langle q \rangle} = \mathbf{dr}_{\langle q \rangle} \cdot \mathbf{dr}_{\langle q \rangle} = g_{ij}\, dq^i\, dq^j \tag{16a}$$

$$ds^2_{\langle q' \rangle} = \mathbf{dr}_{\langle q' \rangle} \cdot \mathbf{dr}_{\langle q' \rangle} = g'_{rs}\, dq'^r\, dq'^s \tag{16b}$$

The interval between two arbitrary points in space is invariant, i.e., $ds^2_{\langle q \rangle} = ds^2_{\langle q' \rangle}$. Using equations (13a), (13b), and (14), we have the equality

$$ds^2 = g_{ij}\, dq^i\, dq^j = g_{ij} \frac{\partial q^i}{\partial q'^r} \frac{\partial q^j}{\partial q'^s}\, dq'^r\, dq'^s \tag{17}$$

Comparing equations (16b) and (17), we obtain

$$g'_{rs} = g_{ij} \frac{\partial q^i}{\partial q'^r} \frac{\partial q^j}{\partial q'^s}, \qquad g_{ij} = g'_{rs} \frac{\partial q'^r}{\partial q^i} \frac{\partial q'^s}{\partial q^j} \tag{18}$$

If we replace the (q')-system with the (x)-system, then $g'_{rs} = \delta_{rs}$ (Kronecker delta) and by equations (18) we have

$$g_{kl} = \delta_{rs} \frac{\partial x^r}{\partial q^k} \frac{\partial x^s}{\partial q^l} = \sum_r \frac{\partial x^r}{\partial q^k} \frac{\partial x^r}{\partial q^l} \tag{19a}$$

$$g_{kl} \frac{\partial q^k}{\partial x^r} \frac{\partial q^l}{\partial x^r} = 1 \qquad \text{or} \qquad \sum_k g_{kk} \left(\frac{\partial q^k}{\partial x^r} \right) = 1 \tag{19b}$$

Let the $\{\mathbf{e}_{qj}\} = (\mathbf{e}_{q1}, \mathbf{e}_{q2}, \mathbf{e}_{q3})$ and $\{\mathbf{e}_{xr}\} = (\mathbf{e}_{x1}, \mathbf{e}_{x2}, \mathbf{e}_{x3})$ be orthogonal unit vectors along the corresponding curve of coordinates which passes through a given point q or x in space. Then we have

$$\mathbf{e}_{qi} \cdot \mathbf{e}_{qj} = \delta_{ij} \qquad \text{or} \qquad \mathbf{e}_{xr} \cdot \mathbf{e}_{xs} = \delta_{rs} \tag{20a}$$

The corresponding $\{\mathbf{e}_q^j\}$ and $\{\mathbf{e}_x^r\}$ are defined as

$$\mathbf{e}_q^i = g^{ij} \mathbf{e}_{qj} \qquad \text{or} \qquad \mathbf{e}_x^r = \delta^{rs} \mathbf{e}_{xs} \tag{20b}$$

Using equation (15), we get

$$g_{ij} \mathbf{e}_q^j = \mathbf{e}_{qi} \qquad \text{and} \qquad g^{ki} \mathbf{e}_{qi} = \mathbf{e}_q^k \tag{20c}$$

The displacement **dr** may be written as

$$\begin{aligned} \mathbf{dr}_{\langle q\rangle} &= \mathbf{e}_{qj}\, dr^j_{\langle q\rangle} = g_{jk}\mathbf{e}^k_q\, dr^j_{\langle q\rangle} = \mathbf{e}^k_q\, dr_{\langle q\rangle k} \\ \mathbf{dr}_{\langle x\rangle} &= \mathbf{e}_{xr}\, dr^r_{\langle x\rangle} = \delta_{rs}\mathbf{e}^s_x\, dr^r_{\langle x\rangle} = \mathbf{e}^s_x\, dr_{\langle x\rangle s} \end{aligned} \tag{21}$$

where

$$dr_{\langle q\rangle k} = g_{jk}\, dr^j_{\langle q\rangle}, \qquad dr_{\langle x\rangle s} = \delta_{rs}\, dr^r_{\langle x\rangle} \tag{22}$$

Hence

$$\begin{aligned} ds^2 &= \mathbf{dr}_{\langle q\rangle}\cdot\mathbf{dr}_{\langle q\rangle} = (\mathbf{e}_{qi}\cdot\mathbf{e}_{qj})\, dr^i_{\langle q\rangle}\, dr^j_{\langle q\rangle} \\ &= \delta_{ij}\frac{\partial r^i_{\langle q\rangle}}{\partial q^k}\frac{\partial r^j_{\langle q\rangle}}{\partial q^l}\, dq^k\, dq^l = g_{kl}\, dq^k\, dq^l \end{aligned} \tag{23a}$$

$$\begin{aligned} ds^2 &= \mathbf{dr}_{\langle x\rangle}\cdot\mathbf{dr}_{\langle x\rangle} = (\mathbf{e}_{xr}\cdot\mathbf{e}_{xs})\, dr^r_{\langle x\rangle}\cdot dr^s_{\langle x\rangle} \\ &= \delta_{rs}\, dr^r_{\langle x\rangle}\cdot dr^s_{\langle x\rangle} = \delta_{rs}\, dx^r\, dx^s \end{aligned} \tag{23b}$$

Now, using equation (14), we can write

$$dr^j_{\langle q\rangle} = \sqrt{g_{ij}}\, dq^j, \qquad dr^r_{\langle x\rangle} = dx^r \tag{24}$$

$g_{jj}\, dq^j$ is the jth component of **dr** relative to the (q)-system and has dimension of length, while the dimension of dq^j is not always length. Using equations (22) and (24), we can write down

$$\begin{aligned} \mathbf{dr}_{\langle q\rangle} &= \mathbf{e}_{qj}\, dr^j_{\langle q\rangle} = \mathbf{e}_{qj}\sqrt{g_{jj}}\, dq^j \\ &= \mathbf{e}^j_q\, dr_{\langle q\rangle j} = \mathbf{e}^j_q\sqrt{g^{jj}}\, dq_j \end{aligned} \tag{25a}$$

$$\begin{aligned} \mathbf{dr}_{\langle x\rangle} &= \mathbf{e}_{xr}\, dr^r_{\langle x\rangle} = \mathbf{e}_{xr}\, dx^r \\ &= \mathbf{e}^r_x\, dr_{\langle x\rangle r} = \mathbf{e}^r_x\, dx_r \end{aligned} \tag{25b}$$

As both $\{q^j\}$ and $\{x^r\}$ are static relative to each other, we have $\mathbf{dr}_{\langle q\rangle} = \mathbf{dr}_{\langle x\rangle}$, i.e.,

$$\begin{aligned} \mathbf{e}_{qj}\sqrt{g_{jj}}\, dq^j &= \mathbf{e}_{xr}\, dx^r = \mathbf{e}_{xr}\frac{\partial x^r}{\partial q^j}\, dq^j \\ \mathbf{e}^j_q\sqrt{g^{jj}}\, dq_j &= \mathbf{e}^r_x\, dx_r = \mathbf{e}^r_x\frac{\partial x_r}{\partial q_j}\, dq_j \end{aligned} \tag{26a}$$

$$\begin{aligned} \mathbf{e}_{xr}\, dx^r &= \mathbf{e}_{qj}\sqrt{g_{jj}}\, dq^j = \mathbf{e}_{qj}\sqrt{g_{jj}}\frac{\partial q^j}{\partial x^r}\, dx^r \\ \mathbf{e}^r_x\, dx_r &= \mathbf{e}^j_q\sqrt{g^{jj}}\, dq_j = \mathbf{e}^j_q\sqrt{g^{jj}}\frac{\partial q_j}{\partial x_r}\, dx_r \end{aligned} \tag{26b}$$

After comparing both sides of equations (25a) and (25b), we get

$$\mathbf{e}_{qj}=\frac{\mathbf{e}_{xr}}{\sqrt{g_{jj}}}\frac{\partial x^r}{\partial q^j} \qquad \text{or} \qquad \mathbf{e}_q^j=\frac{\mathbf{e}_x^r}{\sqrt{g^{jj}}}\frac{\partial x_r}{\partial q_j} \tag{27a}$$

$$\mathbf{e}_{xr}=\mathbf{e}_{qj}\sqrt{g_{jj}}\frac{\partial q^j}{\partial x^r} \qquad \text{or} \qquad \mathbf{e}_x^r=\mathbf{e}_q^j\sqrt{g^{jj}}\frac{\partial q_j}{\partial x_r} \tag{27b}$$

From any of these expressions for $\mathbf{e}_{qj}$ or $\mathbf{e}_q^j$: we can derive the other. For example,

$$\mathbf{e}_q^j=g^{jl}\mathbf{e}_{ql}=g^{jl}\frac{\mathbf{e}_{xr}}{\sqrt{g_{ll}}}\frac{\theta x^r}{\partial q^l}=\frac{1}{g_{jl}}\mathbf{e}_{xr}\frac{\partial x^r}{\partial r^l_{\langle q\rangle}}=\mathbf{e}_{xr}\frac{\partial x^r}{\partial r_{\langle q\rangle j}}$$

$$=\frac{\mathbf{e}_x^r}{\sqrt{g^{jj}}}\frac{\partial x_r}{\partial c_j} \tag{28}$$

By the definition of linear momentum, $\mathbf{P}=\mu\, d\mathbf{r}/dt=\mu\dot{\mathbf{r}}$, and taking into account $d\mathbf{r}_{\langle q\rangle}=d\mathbf{r}_{\langle x\rangle}$, we have $\mathbf{P}_{\langle q\rangle}=\mu\dot{\mathbf{r}}_{\langle q\rangle}=\mu\dot{\mathbf{r}}_{\langle x\rangle}=\mathbf{P}_{\langle x\rangle}$, and

$$\mathbf{P}_{\langle q\rangle}=(\mathbf{e}_{qj}\mu\sqrt{g_{jj}}\,\dot{q}^j)=\mathbf{e}_{qj}\mathbf{P}^j_{\langle q\rangle} \tag{29a}$$

$$\mathbf{P}_{\langle x\rangle}=(\mathbf{e}_{xr}\mu\dot{x}^r)=\mathbf{e}_{xr}\mathbf{P}^r_{\langle x\rangle} \tag{29b}$$

where

$$\begin{aligned} P^j_{\langle q\rangle}&=\mu\sqrt{g_{jj}}\,\dot{q}^j=\mu\dot{r}^j_{\langle q\rangle}=\sqrt{g_{jj}}\,P^j_q \\ P^r_{\langle x\rangle}&=\mu\dot{x}^r=\mu\dot{r}^r_{\langle x\rangle}=P^r_x \end{aligned} \tag{29c}$$

By equations (29a)-(29c), we can write

$$\begin{aligned} \mathbf{P}_{\langle q\rangle}\cdot\mathbf{P}_{\langle q\rangle}&=(\mathbf{e}_{qi}\mu\sqrt{g_{ii}}\,\dot{q}^i)\cdot(\mathbf{e}_{qj}\mu\sqrt{g_{jj}}\,\dot{q}^j)=\delta_{ij}P^i_{\langle q\rangle}P^j_{\langle q\rangle} \\ \mathbf{P}_{\langle x\rangle}\cdot\mathbf{P}_{\langle x\rangle}&=(\mathbf{e}_{xr}\mu\dot{x}^r)\cdot(\mathbf{e}_{xs}\mu\dot{x}^s)=\delta_{rs}P^r_{\langle x\rangle}P^s_{\langle x\rangle} \end{aligned} \tag{29d}$$

In equation (29c), $P^j_q=\mu\dot{q}^j$, $P^r_x=\mu\dot{x}^r$, which are the products of the generalized velocity ($\dot{q}^j$ or $\dot{x}^r$) and the mass μ of the particle, are usually called the components of generalized momentum. The $P^j_{\langle q\rangle}$ and $P^r_{\langle x\rangle}$ are the components of linear momentum in the (q)-system and (x)-system, respectively.

Without loss of the generality, let us consider a single particle moving in a conservative potential. Its Lagrangian is

$$\begin{aligned} L(x,\dot{x})&=\frac{1}{2\mu}\mathbf{P}_{\langle x\rangle}\cdot\mathbf{P}_{\langle x\rangle}-U(x) \\ &=\frac{1}{2\mu}(\mathbf{e}_{xr}\mu\dot{x}^r)(\mathbf{e}_{xs}\mu\dot{x}^s)-U(x) \\ &=\frac{1}{2\mu}\mathbf{P}_{\langle q\rangle}\cdot\mathbf{P}_{\langle q\rangle}-U(q) \\ &=\frac{1}{2\mu}(\mathbf{e}_{qi}\mu\sqrt{g_{ii}}\,\dot{q}^i)\cdot(\mathbf{e}_{qj}\mu\sqrt{g_{jj}}\,\dot{q}^j)-U(q) \end{aligned} \tag{30}$$

and $U(x)=U(q)$ is potential energy. According to definition, the canonical momentum is

$$P_{(q)i}=\frac{\partial L(q,q)}{\partial \dot{q}^i}=\frac{1}{\mu}(\mathbf{e}_{qi}\mu\sqrt{g_{ii}})\cdot(\mathbf{e}_{qj}\mu\sqrt{g_{jj}}\,\dot{q}^j)=\sqrt{g_{ii}}\,P^i_{\langle q\rangle} \qquad (i=1,2,3) \tag{31a}$$

$$P_{(x)r}=\frac{\partial L(x,\dot{x})}{\partial \dot{x}^r}=\frac{1}{\mu}(\mathbf{e}_{xr}\mu)(\mathbf{e}_{xs}\mu\dot{x}^s)=P^r_{\langle x\rangle} \qquad (r=1,2,3) \tag{31b}$$

As shown, due to differences in definition, a moving particle has three types of momentum with respect to any given coordinate system. They are the linear momentum $\mathbf{P}_{\langle q\rangle}=\mu\dot{\mathbf{r}}_{\langle q\rangle}$ (since $\dot{\mathbf{r}}_{\langle q\rangle}=\dot{\mathbf{r}}_{\langle q'\rangle}$, this definition itself may not be related to any coordinate system), the canonical momentum $P_{(q)k}=\partial L/\partial\dot{q}^k$, and the generalized momentum $P_{qk}=\mu\dot{q}^k$ (this is only an abbreviation with no relation with the theoretical structure). From equations (31a) and (31b) we see that the relations among them are

$$P^k_{\langle q\rangle}=\frac{1}{\sqrt{g_{kk}}}P_{(q)k}, \qquad P^k_{\langle q\rangle}=\sqrt{g_{kk}}\,P^k_q \tag{32a}$$

$$P^r_{\langle x\rangle}=P_{(x)r}=P^r_x \tag{32b}$$

It may be seen from equations (32a) and (32b) that the corresponding components of the three types of momenta are equal just in the (x)-system. Generally speaking, in a curvilinear coordinate system, not only may the corresponding components of the three types of momenta have different dimensions, but also different components of a given type of momentum may have different dimensions. We must not demand the operators $\hat{P}^k_{\langle q\rangle}$ and $\hat{P}_{(q)k}$ in any (q)-system take a uniform form.

3. SIMPLEST FORM FOR THE OPERATORS $\hat{P}^k_{\langle q\rangle}$ AND $\hat{P}_{(q)k}$

3.1. The Simplest Form of Operator $\hat{P}_{(q)k}$

We apply the transition procedure of equation (2a) to seek a general and simplest form of the operator $\hat{P}_{(q)k}$ of canonical momentum $P_{(q)k}$ in any orthogonal curvilinear coordinate system.

It is clear that the value of PB for q^k and $P_{(q)l}$ is δ^k_l, i.e.,

$$\{q^k, P_{(q)l}\}=\delta^k_l \tag{33}$$

where $\delta^k_l=1$ (as $k=l$) or 0 (as $k\neq l$). By equation (2a), we have

$$\{q^k, P_{(q)l}\}=\delta^k_l \rightarrow \frac{1}{i\hbar}[\hat{q}^k, \hat{P}_{(q)l}]=\delta^k_l \tag{34}$$

If we confine ourselves only to the condition that the commutation bracket in (34) be valid, we can generally take $\hat{P}_{(q)l}$ in the form

$$\hat{P}_{(q)l} = \frac{\hbar}{i}\frac{\partial}{\partial q^l} + f_l(q) \tag{35}$$

where $f_l(q)$ is an arbitrary real function of q, and can be written as $f_l = \partial\varphi(q)/\partial q^l$, where $\varphi(q)$ is a scalar function. Now let $\hat{q}^k$, $\hat{P}_{(q)l}$ undergo a unitary transformation $\exp[-i\varphi(q)/\hbar]$ to change the gauge for $\hat{P}_{(q)l}$ in (35) and arrive at the simplest form: $\hat{q}'^k$ and $\hat{P}'_{(q)l}$. In the coordinate representation, $\hat{q}^k = q^k$, we have

$$\hat{q}'^k|\rangle = \left\{\exp\left[\frac{i}{\hbar}\varphi(q)\right]\hat{q}^k \exp\left[-\frac{i}{\hbar}\varphi(q)\right]\right\}\Big|\Big\rangle$$
$$= q^k|\rangle = \hat{q}|\rangle \tag{36a}$$

$$\hat{P}'_{(q)l}|\rangle = \left\{\exp\left[\frac{i}{\hbar}\varphi(q)\right]\left(\frac{\hbar}{i}\frac{\partial}{\partial q^l} + \frac{\partial(q)}{\partial q^l}\right)\exp\left[-\frac{i}{\hbar}\varphi(q)\right]\right\}\Big|\Big\rangle$$
$$= \frac{\hbar}{i}\frac{\partial}{\partial q^l}\Big|\Big\rangle \tag{36b}$$

Since $|\rangle$ is arbitrary, we obtain

$$\hat{q}'^l = \hat{q}^l = q^l, \qquad \hat{P}'_{(q)l} = \frac{\hbar}{i}\frac{\partial}{\partial q^l}$$

Thus the unitary transformation $\exp[-i\varphi(q)/\hbar]$ does not change the coordinate operator, but only the momentum operator. Without loss of generality, the operator of the canonical momentum in the (q)-system may always be taken as the following simplest uniform form:

$$\hat{P}_{(q)l} = \frac{\hbar}{i}\frac{\partial}{\partial q^l} \qquad (l = 1, 2, 3) \tag{37}$$

In this view, the canonical momentum operator in equation (2b) is reasonable.

3.2. The Simplest Form of Operator $\hat{P}_{(q)l}$

Although we concluded that all momenta and the corresponding operators in (2a), (2b) are to be canonical momenta rather than linear, from the viewpoint of physical meaning it is required that the momenta be pure linear instead of canonical when we compose them with other physical quantities. [Of course, by equations (32a) and (32b) all the physical quantities containing momentum can also be expressed through the use of the canonical momentum.] It can be seen from this that when we compose quantum operators corresponding to physical quantities containing momentum, the linear momentum operator is much more consistent than the canonical one.

Since the classical Poisson bracket may be composed of any two functions of canonical variables, while the q^k, $P_{\langle q\rangle l}$ in equation (32a) are also functions of canonical variables, we have the following Poisson bracket:

$$\{q^k, P^l_{\langle q\rangle}\} = \left(\frac{\partial q^k}{\partial q^i}\frac{\partial P^l_{\langle q\rangle}}{\partial P_{(q)i}} - \frac{\partial P^l_{\langle q\rangle}}{\partial q^i}\frac{\partial q^k}{\partial P_{(q)i}}\right)$$

$$= \frac{1}{\sqrt{g_{ll}}}\,\delta^k_i\delta^i_l = \frac{1}{\sqrt{g_{ll}}}\,\delta^k_l \tag{38}$$

It can be seen from equations (19a), (19b) that the metric g_{kl} is a function of coordinate variables only. That is, in the coordinate representation, $\hat{g}_{kl} = g_{kl}$. Using the transition procedure (2a), we have

$$\{q^k, P^l_{\langle q\rangle}\} = \frac{1}{\sqrt{g_{ll}}}\,\delta^k_l \xrightarrow{\text{step B}} \frac{1}{i\hbar}[\hat{q}^k, \hat{P}^l_{\langle q\rangle}] = \frac{1}{\sqrt{g_{ll}}}\,\delta^k_l \tag{39}$$

By a similar method, after analyzing the dimension of the commutation bracket in equation (39), we can write the expression for $\hat{P}^l_{\langle q\rangle}$ as

$$\hat{P}^l_{\langle q\rangle} = \frac{\hbar}{i}\frac{1}{\sqrt{g_{ll}}}\frac{\partial}{\partial q^l} + \frac{1}{\partial q^l} + \frac{1}{\sqrt{g_{ll}}}\frac{\partial \varphi(q)}{\partial q^l} \tag{40}$$

and

$$\hat{\mathbf{P}}_{\langle q\rangle} = \frac{\hbar}{i}\frac{\mathbf{e}^l_q}{\sqrt{g_{ll}}}\frac{\partial}{\partial q^l} + \nabla_{\langle q\rangle}\varphi(q) \tag{41}$$

where

$$\nabla_{\langle q\rangle} = \frac{\mathbf{e}^l_q}{\sqrt{g_{ll}}}\frac{\partial}{\partial q^l} \tag{42}$$

is a gradient operator in the (q)-system. Since the interaction of a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ with a charged particle is accomplished by the replacement

$$\mathbf{P} \to \left(\mathbf{P}_{\langle q\rangle} - \frac{e}{c}\mathbf{A}\right) \tag{43}$$

it is clear that $\nabla_{\langle q\rangle}\varphi(q)$ in (41) is similar to a vector potential $-(e/c)\mathbf{A}$. Because $\nabla_{\langle q\rangle} \times \nabla_{\langle q\rangle} = 0$, $\nabla_{\langle q\rangle}\varphi(q)$ may be interpreted as the vector potential of the zero magnetic field. According to the discussion used to establish equation (37), this vector potential may also be removed by an unitary transformation $\exp[-i\varphi(q)/h]$. Hence, the operators of linear momentum in the (q)-system may always be taken as

$$\hat{\mathbf{P}}_{\langle q\rangle}=\frac{\hbar}{i}\frac{\mathbf{e}_q^l}{\sqrt{g_{ll}}}\frac{\partial}{\partial q^l}=\frac{\hbar}{i}\nabla_{\langle q\rangle} \tag{44a}$$

$$\hat{\mathbf{P}}_{\langle x\rangle}=\frac{\hbar}{i}\mathbf{e}_x^r\frac{\partial}{\partial x^r}=\frac{\hbar}{i}\nabla_{\langle x\rangle} \tag{44b}$$

Their components are

$$\hat{P}^l_{\langle q\rangle}=\frac{\hbar}{i}\frac{1}{\sqrt{g_{ll}}}\frac{\partial}{\partial q^l}=\frac{1}{\sqrt{g_{ll}}}\hat{P}_{(q)l} \tag{45a}$$

$$\hat{P}^r_{\langle x\rangle}=\frac{\hbar}{i}\frac{\partial}{\partial x^r}=\hat{P}_{(x)r} \tag{45b}$$

We see from (45a), (45b) that in any coordinate system the component operators of momentum are of two types, which are associated with the components of linear and canonical momenta, respectively. Only in the (x)-system are $\hat{P}^l_{\langle x\rangle}$ and $\hat{P}_{(x)l}$ equal. By comparing equations (32a) and (32b) and equations (45a) and (45b), it can be shown that the relation between the operators of linear and canonical momenta agrees with the corresponding relation between the classical quantities.

4. HAMILTONIAN OPERATOR, REPEAL OF THE (x)-COMPONENT

We prove now that any physical quantity that contains and is expressed by means of $\mathbf{P}_{\langle q\rangle}$ or $P^l_{\langle q\rangle}(1,2,3)$ instead of $\mathbf{P}_{(q)}$ can be transformed directly to a corresponding correct operator in any orthogonal curvilinear coordinate system. In particular, the classical Hamiltonian in any orthogonal (q)-system can be transformed directly to the corresponding correct Hamiltonian operator along step B of the process in Figure 1, i.e., we obtain

$$\hat{H}(\hat{q},\hat{p}_{(q)})=\hat{H}'(\hat{q},\hat{p}_{(q)}) \tag{46}$$

We may say that Figure 1 is closed.

Let the classical quantity $\mathbf{P}_{\langle x\rangle}\cdot\mathbf{P}_{\langle x\rangle}$ undergo successively steps C and B of Figure 1, i.e.,

$$\mathbf{P}_{\langle x\rangle}\cdot\mathbf{P}_{\langle x\rangle}\xrightarrow{\text{step C}}\mathbf{P}_{\langle q\rangle}\cdot\mathbf{P}_{\langle q\rangle}\xrightarrow{\text{step B}}\hat{\mathbf{P}}_{\langle q\rangle}\cdot\hat{\mathbf{P}}_{\langle q\rangle} \tag{47}$$

Since $\mathbf{P}_{\langle x\rangle}=\mathbf{P}_{\langle q\rangle}$, step C here evidently is valid. Now it needs to be proved that the operator $\hat{\mathbf{P}}_{\langle q\rangle}\cdot\hat{\mathbf{P}}_{\langle q\rangle}$, as a result of step B, will present a correct form. For this purpose, substituting the operator of (44a) into the right-hand side of step B in (47), we have

$$\hat{\mathbf{P}}_{\langle q\rangle}\hat{\mathbf{P}}_{\langle q\rangle}=\left(\frac{\hbar}{i}\right)^2\left[\left(\frac{\mathbf{e}_q^k}{\sqrt{g_{kk}}}\frac{\partial}{\partial q^k}\right)\cdot\left(\frac{\mathbf{e}_q^l}{\sqrt{g_{ll}}}\frac{\partial}{\partial q^l}\right)\right]$$

Because the direction of $\mathbf{e}_q^k$ is variable with the point q, using equations (27a), (27b), and (15) and

$$\mathbf{e}_x^r\frac{\partial x_r}{\partial q_k}=\mathbf{e}_{xs}g^{rs}\frac{1}{g_{ki}}\frac{\partial x_r}{\partial q^i}=\mathbf{e}_{xs}g^{ki}\frac{\partial x^s}{\partial q^i}=\mathbf{e}_{xr}g^{ki}\frac{\partial x^r}{\partial q^i}\tag{48}$$

we get

$$\hat{\mathbf{P}}_{\langle q\rangle}\cdot\hat{\mathbf{P}}_{\langle q\rangle}=\left(\frac{\hbar}{i}\right)^2(\mathbf{e}_{xr}\cdot\mathbf{e}_{xs})\left[\left(g^{ki}\frac{\partial x^r}{\partial q^i}\frac{\partial}{\partial q^k}\right)\left(g^{lj}\frac{\partial x^s}{\partial q^j}\frac{\partial}{\partial q^l}\right)\right]\tag{49}$$

Let both sides of equation (49) act on an arbitrary ket $|\rangle$; we obtain

$$\begin{aligned}\hat{\mathbf{P}}_{\langle q\rangle}\hat{\mathbf{P}}_{\langle q\rangle}|\rangle&=\left(\frac{\hbar}{i}\right)^2\delta_{rs}\left[\left(g^{ki}\frac{\partial x^r}{\partial q^i}\frac{\partial}{\partial q^k}\right)\left(g^{lj}\frac{\partial x^s}{\partial q^j}\frac{\partial}{\partial q^l}\right)\right]\Big|\Big\rangle\\&=\left(\frac{\hbar}{i}\right)^2\delta_{rs}\left[\frac{\partial}{\partial q^k}\left(g^{ki}g^{lj}\frac{\partial x^r}{\partial q^i}\frac{\partial x^s}{\partial q^j}\right)\right]\frac{\partial|\rangle}{\partial q^l}\\&\quad-\left(\frac{\hbar}{i}\right)^2\delta_{rs}\left[g^{lj}\frac{\partial x^s}{\partial q^j}\frac{\partial}{\partial q^k}\left(g^{ki}\frac{\partial x^r}{\partial q^i}\right)\right]\frac{\partial|\rangle}{\partial q^l}\\&\quad+\left(\frac{\hbar}{i}\right)^2\delta_{rs}\left[g^{ki}g^{lj}\left(\frac{\partial x^r}{\partial q^i}\right)\left(\frac{\partial x^s}{\partial q^j}\right)\right]\frac{\partial^2|\rangle}{\partial q^k\partial q^l}\end{aligned}\tag{50}$$

Using equations (19a), (15), and (14), we can write the first term of equation (50) as

$$\begin{aligned}\left(\frac{\hbar}{i}\right)^2\delta_{rs}\left[\frac{\partial}{\partial q^k}(g^{ki}g^{lj}g_{ij})\right]\frac{\partial|\rangle}{\partial q^l}&=\left(\frac{\hbar}{i}\right)^2\left[\frac{\partial}{\partial q^k}(g^{ki}\delta_i^l)\right]\frac{\partial|\rangle}{\partial q^l}\\&=\left(\frac{\hbar}{i}\right)^2\sum_k\left(\frac{\partial g^{kk}}{\partial q^k}\frac{\partial|\rangle}{\partial q^k}\right)\\&=\left(\frac{\hbar}{i}\right)^2\sum_k\left(-\frac{i}{g_{kk}^2}\frac{\partial g_{kk}}{\partial q^k}\frac{\partial|\rangle}{\partial q^k}\right)\end{aligned}\tag{51}$$

The second term of equation (50) can be written as

$$-\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{lj}\left(\frac{\partial x^s}{\partial q^j}\right)\frac{\partial}{\partial q^k}\left(g^{ki}\frac{\partial x^r}{\partial q^i}\right)\right]\frac{\partial|\rangle}{\partial q^l}$$

$$=-\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{lj}\frac{\partial}{\partial q^k}\left(g^{ki}\frac{\partial x^r}{\partial q^i}\frac{\partial x^s}{\partial q^j}\right)\right]\frac{\partial|\rangle}{\partial q^l}$$

$$+\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{ki}g^{lj}\frac{\partial x^r}{\partial q^i}\frac{\partial}{\partial q^k}\left(\frac{\partial x^s}{\partial q^j}\right)\right]\frac{\partial|\rangle}{\partial q^l}$$

$$=0+\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{ki}g^{lj}\frac{\partial x^r}{\partial q^i}\frac{\partial}{\partial q^k}\left(\frac{\partial x^s}{\partial q^j}\right)\right]\frac{\partial|\rangle}{\partial q^l} \tag{52a}$$

After exchanging the symbols $k \rightleftharpoons l$ and $i \rightleftharpoons j$, according as $k=l$ and $k\neq l$, the second term is divided into two parts. Next, using (14), we have

$$\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{ki}g^{lj}\frac{\partial x^r}{\partial q^i}\frac{\partial}{\partial q^k}\left(\frac{\partial x^s}{\partial q^j}\right)\right]\frac{\partial|\rangle}{\partial q^l}$$

$$=\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{ki}g^{lj}\frac{\partial x^s}{\partial q^j}\frac{\partial}{\partial q^l}\left(\frac{\partial x^r}{\partial q^i}\right)\right]\frac{\partial|\rangle}{\partial q^k}$$

$$=\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{ki}g^{lj}\frac{\partial x^s}{\partial q^j}\frac{\partial}{\partial q^l}\left(\frac{\partial x^r}{\partial q^i}\right)\right]_{k=l}\frac{\partial|\rangle}{\partial q^k}$$

$$+\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left[g^{ki}g^{lj}\frac{\partial x^s}{\partial q^j}\frac{\partial}{\partial q^l}\left(\frac{\partial x^r}{\partial q^i}\right)\right]_{k\neq l}\frac{\partial|\rangle}{\partial q^k}$$

$$=\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\sum_k\left(g^{kk}g^{kk}\frac{\partial x^s}{\partial q^k}\frac{\partial}{\partial q^k}\left(\frac{\partial x^r}{\partial q^k}\right)\right)\frac{\partial|\rangle}{\partial q^k}$$

$$+\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\sum_{k,l}\left[g^{kk}g^{ll}\frac{\partial x^s}{\partial q^l}\frac{\partial}{\partial q^l}\left(\frac{\partial x^r}{\partial q^k}\right)\right]_{k\neq l}\frac{\partial|\rangle}{\partial q^k}$$

$$=\left(\frac{\hbar}{i}\right)^2 \sum_k\left(\frac{1}{2g_{kk}^2}\frac{\partial g_{kk}}{\partial q^k}\right)\frac{\partial|\rangle}{\partial q^k}$$

$$+\left(\frac{\hbar}{i}\right)^2 \sum_{k,l}\left(\frac{1}{2g_{kk}g_{ll}}\frac{\partial g_{ll}}{\partial q^k}\right)\frac{\partial|\rangle}{\partial q^k} \tag{52b}$$

Using equation (14), we find that only the third term of equation (50) remains for $k=l$:

$$\left(\frac{\hbar}{i}\right)^2 \delta_{rs}\left(g^{kl}g^{lj}\left(\frac{\partial x^r}{\partial q^i}\right)\left(\frac{\partial x^s}{\partial q^j}\right)\right)\frac{\partial^2|\rangle}{\partial q^k\partial q^l}=\left(\frac{\hbar}{i}\right)^2(g^{ki}g^{lj}g_{ij})\frac{\partial^2|\rangle}{\partial q^k\partial q^l}$$

$$=\left(\frac{\hbar}{i}\right)^2\sum_k\left(\frac{1}{g_{kk}}\frac{\partial^2|\rangle}{\partial q^k\partial q^l}\right) \tag{53}$$

Putting equations (51), (52b), and (53) into equation (50), we obtain

$$\hat{\mathbf{P}}_{\langle q\rangle}\cdot\hat{\mathbf{P}}_{\langle q\rangle}|\rangle=\left(\frac{\hbar}{i}\right)^2\sum_k\left(-\frac{1}{2g_{kk}^2}\frac{\partial g_{kk}}{\partial q^k}\frac{\partial|\rangle}{\partial q^k}\right.$$
$$\left.+\sum_l\left(\frac{1}{2g_{kk}g_{ll}}\frac{\partial g_{ll}}{\partial q^k}\frac{\partial|\rangle}{\partial q^k}\right)_{k\neq l}+\frac{1}{g_{kk}}\frac{\partial^2|\rangle}{\partial q^k\partial q^l}\right)\Bigg|\Bigg\rangle$$
$$=\left(\frac{\hbar}{i}\right)^2\left[\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^1}\left(\frac{\sqrt{g_{(3)}}}{g_{11}}\frac{\partial}{\partial q^1}\right)+\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^2}\left(\frac{\sqrt{g_{(3)}}}{g_{22}}\frac{\partial}{\partial q^2}\right)\right.$$
$$\left.+\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^3}\left(\frac{\sqrt{g_{(3)}}}{g_{33}}\frac{\partial}{\partial q^3}\right)\right]\Bigg|\Bigg\rangle$$
$$=\sum_k\frac{1}{\sqrt{g_{(3)}}}\hat{P}_{(q)k}\left(\frac{\sqrt{g_{(3)}}}{g_{kk}}\hat{P}_{(q)k}\right) \tag{54}$$

where $g_{(3)}=g_{11}g_{22}g_{33}$. Since the ket vector $|\rangle$ is arbitrary, the operator $\hat{\mathbf{P}}_{\langle q\rangle}\cdot\hat{\mathbf{P}}_{\langle q\rangle}$ corresponding to $\mathbf{P}_{\langle q\rangle}\cdot\mathbf{P}_{\langle q\rangle}$ undergoes successively steps C and B to give

$$\hat{\mathbf{P}}_{\langle q\rangle}\cdot\hat{\mathbf{P}}_{\langle q\rangle}=\sum_k\frac{1}{\sqrt{g_{(3)}}}\hat{P}_{(q)k}\left(\frac{\sqrt{g_{(3)}}}{g_{kk}}\hat{P}_{(q)k}\right) \tag{55}$$

In addition to steps C and B in Figure 1, we may also choose the process of letting the classical quantity $\mathbf{P}_{\langle x\rangle}\cdot\mathbf{P}_{\langle x\rangle}$ undergo steps A and D:

$$\mathbf{P}_{\langle x\rangle}\cdot\mathbf{P}_{\langle x\rangle}\xrightarrow{\text{step A}}\hat{\mathbf{P}}_{\langle x\rangle}\cdot\hat{\mathbf{P}}_{\langle x\rangle}\xrightarrow{\text{step D}}\hat{\mathbf{P}}'_{\langle q\rangle}\cdot\hat{\mathbf{P}}'_{\langle q\rangle} \tag{56}$$

Because step A here is performed in the (x)-system, it is clearly valid. Step D is familiar to us and leads to the following operator by the use of the rule of the compound differential:

$$\hat{\mathbf{P}}'_{\langle q\rangle}\cdot\hat{\mathbf{P}}'_{\langle q\rangle}=\sum_k\frac{1}{\sqrt{g_{(3)}}}\hat{P}_{(q)k}\left(\frac{\sqrt{g_{(3)}}}{g_{kk}}\hat{P}_{(q)k}\right) \tag{57}$$

The results (55) and (57) are in agreement with each other, so that the paths in Figure 1 are closed. That is, if the classical Hamiltonian is expressed in terms of linear momentum, the (x)-condition is then unnecessary for determing the Hamiltonian operator. The same conclusion is valid for other classical quantities containing momentum.

For example, the operator of the angular momentum in a spherical coordinate system can be given correctly by the use of the methods discussed above. In fact, the radius vector in a spherical system, abbreviated as (s)-system, may be written as $\mathbf{r}_{\langle s\rangle}=\mathbf{e}_{sr}r$, and the square of the arc element is

$$ds^2_{\langle s\rangle}=\sum_k g_{kk}dq_k^2=dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2 \tag{58}$$

Hence the metric components are

$$g_{11}=1, \qquad g_{22}=r^2, \qquad g_{33}=r^2\sin^2\theta \tag{59}$$

By equations (29a) and (32a), the linear momentum in the (s)-system may be written as

$$\begin{aligned}\mathbf{P}_{\langle s\rangle}&=\mathbf{e}_{sk}P^k_{\langle s\rangle}\\&=\mathbf{e}_{sk}(\mu\sqrt{g_{kk}}\,\dot{q}^k)\\&=\mathbf{e}_{sr}\mu\dot{r}+\mathbf{e}_{s\theta}\mu r\dot{\theta}+\mathbf{e}_{s\varphi}\mu r\sin\theta\dot{\varphi}\\&=\mathbf{e}_{sr}P^r_{\langle s\rangle}+\mathbf{e}_{s\theta}P^\theta_{\langle s\rangle}+\mathbf{e}_{s\varphi}P^\varphi_{\langle s\rangle}\end{aligned} \tag{60}$$

Then the angular momentum and its square in the (s)-system are, respectively,

$$\begin{aligned}\mathbf{L}_{\langle s\rangle}&=\mathbf{r}_{\langle s\rangle}\times\mathbf{P}_{\langle s\rangle}\\&=\mathbf{e}_{sr}\times(\mathbf{e}_{sr}\mathbf{P}^r_{\langle s\rangle}+\mathbf{e}_{s\theta}P^\theta_{\langle s\rangle}+\mathbf{e}_{s\varphi}P^\varphi_{\langle s\rangle}\\&=r(\mathbf{e}_{s\varphi}P^\theta_{\langle s\rangle}-\mathbf{e}_{s\theta}P^\varphi_{\langle s\rangle})\end{aligned} \tag{61}$$

$$L^2_{\langle s\rangle}=\mathbf{L}_{\langle s\rangle}\cdot\mathbf{L}_{\langle s\rangle}=r^2(P^{\theta 2}_{\langle s\rangle}+P^{\varphi 2}_{\langle s\rangle}) \tag{62}$$

Both $\mathbf{L}_{\langle s\rangle}$ and $\mathbf{L}^2_{\langle s\rangle}$ have been expressed in terms of linear momentum in the (s)-system and may directly be transformed to the corresponding operators in the (s)-system by the use of (44a), and the kth dimensional formulas of (55) and (59),

$$\begin{aligned}\hat{\mathbf{L}}_{\langle s\rangle}&=\hat{\mathbf{r}}_{\langle s\rangle}\times\hat{\mathbf{P}}_{\langle s\rangle}\\&=r(\mathbf{e}_{s\varphi}\hat{P}^\theta_{\langle s\rangle}-\mathbf{e}_{s\theta}\hat{P}_{\langle s\rangle})\\&=\left(\frac{\hbar}{i}\right)r\left(\mathbf{e}_{s\varphi}\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{\mathbf{e}_{s\theta}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\\&=\left(\frac{\hbar}{i}\right)\left(\mathbf{e}_{s\varphi}\frac{\partial}{\partial\theta}-\frac{\mathbf{e}_{s\theta}}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\end{aligned} \tag{63}$$

$$\hat{L}^2_{\langle s\rangle}=\left(\frac{\hbar}{i}\right)^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \tag{64}$$

By equations (27a) and (59), we can find the relations between $(\mathbf{e}_{x1}, \mathbf{e}_{x2}, \mathbf{e}_{x3})$ in the (x)-system and $(\mathbf{e}_{qr}, \mathbf{e}_{q\theta}, \mathbf{e}_{q\varphi})$ in the (s)-system,

$$\mathbf{e}_{sr}=\mathbf{e}_{x1}\sin\theta\cos\varphi+\mathbf{e}_{x2}\sin\theta\sin\varphi+\mathbf{e}_{x3}\cos\theta \tag{65a}$$

$$\mathbf{e}_{s\theta}=\mathbf{e}_{x1}\cos\theta\cos\varphi+\mathbf{e}_{x2}\cos\theta\sin\varphi-\mathbf{e}_{x3}\sin\theta \tag{65b}$$

$$\mathbf{e}_{s\varphi}=-\mathbf{e}_{x1}\sin\varphi+\mathbf{e}_{x2}\cos\varphi \tag{65c}$$

Using equation (63), we obtain

$$\hat{L}^1_{\langle x\rangle} = \mathbf{e}_{x1} \cdot \hat{\mathbf{L}}_{\langle s\rangle} = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \tag{66a}$$

$$\hat{L}^2_{\langle x\rangle} = \mathbf{e}_{x2} \cdot \hat{\mathbf{L}}_{\langle s\rangle} = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right) \tag{66b}$$

$$\hat{L}^3_{\langle x\rangle} = \mathbf{e}_{x3}\hat{\mathbf{L}}_{\langle s\rangle} = -i\hbar\frac{\partial}{\partial\varphi} \tag{66c}$$

As we can see, in the (s)-system the operators $\hat{\mathbf{L}}_{\langle s\rangle}$, $\hat{L}^2_{\langle s\rangle}$, and $\hat{L}^1_{\langle x\rangle}$, $\hat{L}^2_{\langle x\rangle}$, $\hat{L}^3_{\langle x\rangle}$ are in agreement with those in the (x)-system; thus, the restriction of the (x)-condition can also be removed for the operator substitution of the angular momentum.

5. REPEAL OF SYMMETRIZATION PROCEDURE

If two classical quantities that are canonical conjugate to each other are contained in any term of a classical equality, their order is ambiguous. In order for such terms to be transformed to a Hermitian operator, the symmetrization procedure seems necessary. In fact, such types of classical equalities include the transformation expressions between linear momentum components and canonical ones as caused by a change of coordinate systems.

Suppose that the (q')-system in (13a), (13b) is replaced by the (x)-system; we have

$$q^k = q^k(x^1, x^2, x^3) = q^k(x), \qquad k = 1, 2, 3 \tag{67a}$$

$$x^r = x^r(q^1, q^2, q^3) = x^r(q), \qquad r = 1, 2, 3 \tag{67b}$$

Taking the derivative of equations (67a), (67b) with respect to t and the generalized velocity $\dot{q}^k$ or $\dot{x}^r$ successively, we get

$$\frac{\partial\dot{q}}{\partial\dot{x}} = \frac{\partial q}{\partial x} \qquad \text{and} \qquad \frac{\partial\dot{x}}{\partial\dot{q}} = \frac{\partial x}{\partial q} \tag{68}$$

Since the (q)-system and (x)-system are static relative to each other, the Lagrangian of a given mechanical system is independent of the choice of system, i.e., $L(q, \dot{q}) = L(x, \dot{x})$. Hence we get

$$P_{(q)k} = \frac{\partial L(q, \dot{q})}{\partial\dot{q}^k} = \frac{\partial\dot{x}^r}{\partial\dot{q}^k}\frac{\partial L(x, \dot{x})}{\partial\dot{x}^r} = \frac{\partial x^r}{\partial q^k}P_{(x)r} \tag{69a}$$

$$P_{(x)r} = \frac{\partial L(x, \dot{x})}{\partial\dot{x}^r} = \frac{\partial\dot{q}^k}{\partial\dot{x}^r}\frac{\partial L(q, \dot{q})}{\partial\dot{q}^k} = \frac{\partial q^k}{\partial x^r}P_{(q)k} \tag{69b}$$

By the definition of the components of the linear momentum, we have

$$P^k_{\langle q\rangle} = \mu \dot{r}^k_{\langle q\rangle} = \mu\sqrt{g_{kk}}\,\dot{q}^k = \mu\sqrt{g_{kk}}\,\frac{\partial q^k}{\partial x^r}\,\dot{x}^r = \sqrt{g_{kk}}\,\frac{\partial q^k}{\partial x^r}\,P^r_{\langle x\rangle} \tag{70a}$$

$$P^r_{\langle x\rangle} = \mu \dot{r}^r_{\langle x\rangle} = \mu \dot{x}^r = \mu\frac{\partial x^r}{\partial q^k}\,\dot{q}^k = \frac{1}{\sqrt{g_{kk}}}\,\frac{\partial x^r}{\partial q^k}\,P^k_{\langle q\rangle} \tag{70b}$$

Equations (70a) and (70b) are the transformation expressions between the linear momentum components. Using equation (19b), we can prove that there is no conflict between equations (69a), (69b), and equations (70a), (70b).

It can be proved that the form of PB will be invariable when one coordinate system is changed into the other. Suppose $F(q, p_{(q)})$ and $G(q, p_{(q)})$ are two arbitrary scalar functions of the canonical variables (q, p); since

$$F(q, p_{(q)}) = F(x, p_{(x)}), \qquad G(q, p_{(q)}) = G(x, p_{(x)})$$

we have

$$\begin{aligned}\{F(x, p_{(x)}), G(x, p_{(x)})\} &= \left(\frac{\partial F(x, p_{(x)})}{\partial x^i}\,\frac{\partial G(x, p_{(x)})}{\partial p_{(x)i}} - \frac{\partial G(x, p_{(x)})}{\partial x^i}\,\frac{\partial F(x, p_{(x)})}{\partial p_{(x)i}}\right)\\ &= \left(\frac{\partial q^k}{\partial x^i}\,\frac{\partial F(q, p_{(q)})}{\partial q^k}\,\frac{\partial p_{(q)l}}{\partial p_{(x)i}}\,\frac{\partial G(q, p_{(q)})}{\partial p_{(q)l}} - \frac{\partial q^k}{\partial x^i}\,\frac{\partial G(q, p_{(q)})}{\partial q^k}\,\frac{\partial p_{(q)l}}{\partial p_{(x)i}}\,\frac{\partial F(q, p_{(q)})}{\partial p_{(q)l}}\right)\\ &= \frac{\partial q^k}{\partial x^i}\,\frac{\partial x^i}{\partial q^l}\{F(q, p_{(q)}), G(q, p_{(q)})\}\\ &= \{F(q, p_{(q)}), G(q, p_{(q)})\}\end{aligned} \tag{71}$$

In the third step we have used equation (69a).

It is easy to see that the commutator bracket itself, $(1/i\hbar)[A, B]$, also is a Hermitian operator provided each of A and B is a Hermitian operator.

Now, let us consider the situation that equation (69a) is substituted into the corresponding operator expression without the symmetrization procedure. Let the classical momentum $P_{(x)i}$ undergo successively steps C and B of Figure 1 and represent the resulting operator in the (q)-system as $\hat{Q}(\hat{q}, \hat{p}_{(q)})$, and then let the same $P_{(x)i}$ undergo steps A and D of Figure 1 with the resulting operator denoted as $\hat{Q}'(\hat{q}, \hat{p}_{(q)})$. Obviously, if the operator

$\hat{Q}(\hat{q}, \hat{p}_{(q)})$ is equal to $\hat{Q}'(\hat{q}, \hat{p}_{(q)})$, and they are Hermitian operators, then both the symmetrization procedure and the (x)-condition are unnecessary.

Let the canonical momentum component $P_{(x)i}$ undergo step C of Figure 1. Taking into account equation (71), we have

$$P_{(x)i} = \mu\{x^i, H(x, p_{(x)})\} \xrightarrow{\text{step C}} \mu\{x^i(q), H(q, p_{(q)})\} \tag{72a}$$

It can be proved that the right-hand side of (72a) is equal to the right-hand side of (69b). We have

$$\begin{aligned}
P_{(x)i} &= \mu\{x^i(q), H(q, p_{(q)})\} \\
&= \mu\left(\frac{\partial x^i(q)}{\partial q^k}\frac{\partial H(q, p_{(q)})}{\partial p_{(q)k}} - \frac{\partial H(q, p_{(q)})}{\partial q^k}\frac{\partial x^i(q)}{\partial p_{(q)k}}\right) \\
&= \mu\frac{\partial x^i(q)}{\partial q^k}\frac{\partial H(q, p_{(q)})}{\partial p_{(q)k}} \\
&= \frac{\partial x^i(q)}{\partial q^k}\dot{q}^k \\
&= \frac{\partial x^i(q)}{\partial q^k}\frac{1}{\sqrt{g_{kk}}}P^k_{\langle q\rangle} \\
&= \sum_k \frac{1}{g_{kk}}\frac{\partial x^i(q)}{\partial q^k}P_{(q)k} \\
&= g^{jk}\frac{\partial x^i(q)}{\partial q^j}P_{(q)k} \qquad (\text{for } j \neq k,\ g^{jk} = 0)
\end{aligned}$$

According to (19a),

$$g_{jk} = \sum_s \frac{\partial x^s}{\partial q^j}\frac{\partial x^s}{\partial q^k}$$

Hence we have

$$g^{jk} = \sum_s \frac{\partial q^j}{\partial x^s}\frac{\partial q^k}{\partial x^s} \tag{73}$$

$$g^{jk}\frac{\partial x^i}{\partial q^j} = \sum_s \frac{\partial q^j}{\partial x^s}\frac{\partial q^k}{\partial x^s}\frac{\partial x^i}{\partial q^j} = \frac{\partial q^k}{\partial x^i} \tag{74}$$

and

$$P_{(x)i} = g^{jk}\frac{\partial x^i(q)}{\partial q^j}P_{(q)k} = \frac{\partial q^k}{\partial x^i}P_{(q)k} \tag{72b}$$

We can see that the result of taking step C in equation (72a) is in agreement with equation (69b) given by the method of the compound differential.

Now, perform the transition procedure (2a) and let the right-hand side of (72a) undergo step B of Figure 1:

$$P_{(x)i}=\mu\{x^i(q),H(q,p_{(q)})\}\xrightarrow{\text{step B}}\frac{\mu}{i\hbar}[\hat{x}^i(\hat{q}),\hat{H}(\hat{q},\hat{p}_{(q)})] \qquad (75)$$

By equation (55), the Hamiltonian operator in the (q)-system is

$$\hat{H}(\hat{q},\hat{p}_{(q)})=\frac{1}{2\mu}\frac{1}{\sqrt{g_{(3)}}}\hat{P}_{(q)k}\left(\frac{\sqrt{g_{(3)}}}{g_{kl}}\hat{P}_{(q)l}\right)+\hat{U}(\hat{q}) \qquad (76)$$

Choosing the coordinate representation $[\hat{x}^i(q),\hat{U}(\hat{q})]=0$, substituting equation (76) into the right-hand side of (75), and letting it act on an arbitrary ket vector, we get

$$\begin{aligned}
&\frac{\mu}{i\hbar}[\hat{x}^i(\hat{q}),\hat{H}(\hat{q},\hat{p}_{(q)})]|\rangle\\
&=\frac{1}{2i\hbar}\left\{x^i(q)\frac{1}{\sqrt{g_{(3)}}}\hat{P}_{(q)k}(\sqrt{g_{(3)}}\,g^{kl}\hat{P}_{(q)l})\right.\\
&\qquad\left.-\frac{1}{\sqrt{g_{(3)}}}\hat{P}_{(q)k}(\sqrt{g_{(3)}}\,g^{kl}\hat{P}_{(q)l})x^i(q)\right\}\Big|\Big\rangle\\
&=\frac{i\hbar}{2}\left\{x^i(q)\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^k}\left(\sqrt{g_{(3)}}\,g^{kl}\frac{\partial}{\partial q^l}\right)\right.\\
&\qquad\left.-\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^k}\left(\sqrt{g_{(3)}}\,g^{kl}\frac{\partial}{\partial q^l}\right)x^i(q)\right\}\Big|\Big\rangle\\
&=\frac{i\hbar}{2}\left\{x^i(q)\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^k}\left(\sqrt{g_{(3)}}\,g^{kl}\frac{\partial|\rangle}{\partial q^l}\right)\right.\\
&\qquad-\left[\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^k}\left(\sqrt{g_{(3)}}\,g^{kl}\frac{\partial x^i(q)}{\partial q^l}\right)\right]\Big|\Big\rangle\,g^{kl}\frac{\partial x^i(q)}{\partial q^l}\frac{\partial|\rangle}{\partial q^k}\\
&\qquad\left.-g^{kl}\frac{\partial x^i(q)}{\partial q^k}\frac{\partial|\rangle}{\partial q^l}-x^i(q)\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^k}\left(\sqrt{g_{(3)}}\,g^{kl}\frac{\partial|\rangle}{\partial q^l}\right)\right\}
\end{aligned}$$

The summation for the first and fifth terms is equal to zero. Exchanging k and l in the third term, we obtain

$$\begin{aligned}
\frac{\mu}{ih}[\hat{x}^i(\hat{q}),\hat{H}(\hat{q},\hat{p}_{(q)})]|\rangle=\frac{\hbar}{2i}&\left\{2g^{kl}\frac{\partial x^i(q)}{\partial q^k}\frac{\partial}{\partial q^l}\right.\\
&\left.+\frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^k}\left(\sqrt{g_{(3)}}\,g^{kl}\frac{\partial x^i(q)}{\partial q^l}\right)\right\}\Big|\Big\rangle \qquad (77)
\end{aligned}$$

Since $|\rangle$ is arbitrary, we have

$$\frac{\mu}{i\hbar}[\hat{x}^i(\hat{q}), \hat{H}(\hat{q}, \hat{p}_{(q)})] = \frac{\hbar}{2i}\left\{2g^{kl}\frac{\partial x^i(q)}{\partial q^k}\frac{\partial}{\partial q^l} + \frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial q^k}\left(\sqrt{g_{(3)}}\, g^{kl}\frac{\partial x^i(q)}{\partial q^l}\right)\right\} \tag{78}$$

The second term in equation (78) is a scalar function $x^i(q)$ acted on by the Laplacian in the (q)-system. Since $\nabla^2_{\langle q\rangle} = \nabla^2_{\langle x\rangle}$, while $\nabla^2_{\langle x\rangle} x^i = 0$, the second term of (78) is equal to zero. Putting equation (74) into the first term of the right-hand side of equation (78), we obtain

$$\frac{\mu}{ih}[\hat{x}^i(\hat{q}), \hat{H}(\hat{q}, \hat{p}_{(q)})] = \frac{\partial q^l}{\partial x^i}\hat{P}_{(q)l} \tag{79}$$

Because $\hat{x}^i(\hat{q})$ and $\hat{H}(\hat{q}, \hat{p}_{(q)})$ are Hermitian operators, both sides of (79) are also Hermitian operators. On the other hand, let the canonical momentum component $P_{(q)i}$ undergo step A of Figure 1 and use equation (37); then we have

$$P_{(x)i} \xrightarrow{\text{step A}} \hat{P}_{(x)i} = \frac{\hbar}{i}\frac{\partial}{\partial x^i} \tag{80}$$

With the help of the rule of compound differentiation, let the right-hand side of equation (80) undergo step D of Figure 1 and use equation (37); then we obtain

$$\hat{P}_{(x)i} = \frac{\hbar}{i}\frac{\partial}{\partial x^i} \xrightarrow{\text{step D}} \frac{\hbar}{i}\frac{\partial q^k}{\partial x^i}\frac{\partial}{\partial q^k} = \frac{\partial q^k}{\partial x^i}\hat{P}_{(q)k} \tag{81a}$$

The operator of the right side here agrees with that of equation (79). This means that Figure 1 is closed.

Let $(\partial x^i/\partial q^j)$ left-multiply both sides of equation (81a) and sum over the index i; then we have

$$\frac{\partial x^i}{\partial q^l}\hat{P}_{(x)i} = \frac{\partial x^i}{\partial q^l}\frac{\partial q^k}{\partial x^i}\hat{P}_{(q)k} = \hat{P}_{(q)l} \tag{81b}$$

Equations (81a) and (81b) are homologues of equations (69a) and (69b), respectively.

We have seen from the discussion that not only is the transformation expression a Hermitian expression, but also both the restrictions of the (x)-condition and the symmetrization procedure have been removed.

The transformation expression between linear momentum components can be derived by the use of (45). From equation (81b), we can write

$$\sqrt{g_{kk}}\,\hat{P}^k_{\langle q\rangle}=\hat{P}_{(q)k}=\frac{\partial x^i}{\partial q^k}\hat{P}_{(x)i}=\delta_{ij}\frac{\partial x^i}{\partial q^k}P^j_{\langle x\rangle}$$

or

$$\hat{P}^k_{\langle q\rangle}=\frac{\delta_{ij}}{\sqrt{g_{kk}}}\frac{\partial x^i}{\partial q^k}\hat{P}^j_{\langle x\rangle} \tag{82}$$

Inserting equation (19b) in the right-hand side of the above equation and using equation (14), we get

$$\hat{P}^k_{\langle q\rangle}=\frac{\delta_{ij}}{\sqrt{g_{kk}}}\left(\frac{\partial x^i}{\partial q^k}\right)g_{rs}\left(\frac{\partial q^r}{\partial x^i}\frac{\partial q^s}{\partial x^i}\right)\hat{P}^j_{\langle x\rangle}=\sqrt{g_{kk}}\left(\frac{\partial q^k}{\partial x^i}\right)\hat{P}^i_{\langle x\rangle} \tag{83a}$$

Using equation (81a), we obtain

$$\begin{aligned}\hat{P}^i_{\langle x\rangle}&=\hat{P}_{(x)i}\\&=\frac{\partial q^k}{\partial x^i}\hat{P}_{(q)k}\\&=\delta_{kl}\sqrt{g_{kk}}\frac{\partial q^k}{\partial x^i}\hat{P}^l_{\langle q\rangle}\\&=\delta_{kl}\sqrt{g_{kk}}\frac{\partial q^k}{\partial x^i}g^{rs}\frac{\partial x^i}{\partial q^r}\frac{\partial x^i}{\partial q^s}\hat{P}^l_{\langle q\rangle}\\&=\delta_{kl}\delta_{kr}(g_{kk}g^{rs}g^{rs})^{1/2}\frac{\partial x^i}{\partial q^s}\hat{P}^l_{\langle q\rangle}\\&=\sqrt{g^{kk}}\frac{\partial x^i}{\partial q^k}\hat{P}^k_{\langle q\rangle}\\&=\frac{1}{\sqrt{g_{kk}}}\frac{\partial x^i}{\partial q^k}\hat{P}^k_{\langle q\rangle}\end{aligned} \tag{83b}$$

Equations (83a) and (83b) are homologues of equations (70a) and (70b), respectively.

In addition to the above discussions, we must inquire how, in an arbitrary curvilinear coordinate system, the general expression that includes the classical momentum can go over into a Hermitian expression of corresponding operators, in terms of the excluded (x)-condition and symmetrization procedure.

For example, suppose that an equality which includes the linear momentum components is

$$\mathbf{e}_{qk}\varphi_l^k(q)P_{\langle q\rangle}^l = \mathbf{e}_{xr}\psi_s^r(x)P_{\langle x\rangle}^s \tag{84}$$

where $\varphi_l^k(q)$ and $\psi_s^r(x)$ are arbitrary functions of q and x, respectively. Is it necessary that we inquire into the operator expression corresponding to the above equality?

Evidently, for equation (84) to have the necessary physical meaning, it must not be in conflict with the following equality:

$$\mathbf{e}_{qj}P_{\langle q\rangle}^j = \mathbf{e}_{xr}P_{\langle x\rangle}^r \tag{85}$$

Using the $\mathbf{e}_{qj}$ dot product of equation (84), we obtain

$$\mathbf{e}_{qj}\cdot\mathbf{e}_{qk}\varphi_l^k(q)P_{\langle q\rangle}^l = \varphi_l^j(q)P_{\langle q\rangle}^l = \mathbf{e}_{qj}\cdot\mathbf{e}_{xr}\psi_s^r(x)P_{\langle x\rangle}^s$$

or

$$P_{\langle q\rangle}^l = \sum_j (\mathbf{e}_{qj}\cdot\mathbf{e}_{xr})\frac{\psi_s^r(x)}{\varphi_l^j(q)}P_{\langle x\rangle}^s \tag{86}$$

Using the $\mathbf{e}_{ql}$ dot product of equation (85), we obtain

$$\mathbf{e}_{ql}\cdot\mathbf{e}_{qj}P_{\langle q\rangle}^j = (\mathbf{e}_{ql}\cdot\mathbf{e}_{xr})P_{\langle x\rangle}^r$$

or

$$P_{\langle q\rangle}^l = (\mathbf{e}_{ql}\cdot\mathbf{e}_{xr})P_{\langle x\rangle}^r \tag{87}$$

Since equation (84) must be consistent with equation (85), we will have to require that equations (86) and (87) are in direct proportion or simply equal to each other, i.e.,

$$(\mathbf{e}_{ql}\cdot\mathbf{e}_{xr})P_{\langle x\rangle}^r = \sum_j (\mathbf{e}_{qj}\mathbf{e}_{xr})\frac{\psi_s^r(x)}{\varphi_l^j(q)}P_{\langle x\rangle}^s$$

The different components of momentum are independent of each other; we have to set $j=1$, $r=s$, and

$$\varphi_j^j(q) = \psi_r^r(x) = C \tag{88}$$

where C is an arbitrary real constant. It is clear that the problem of replacing equation (86) by a corresponding operator expression returns to the same thesis of equations (70a) and (70b). With similar reasoning, other equalities containing the linear or canonical momentum must not conflict with equations (69a) and (69b), and equations (70a) and (70b), in order to keep the necessary physical meaning. Therefore, it is unnecessary to discuss the requirement of the (x)-condition and the symmetrization procedure.

6. CONCLUSION

First, since both the classical Poisson bracket and the quantum commutation bracket have completely similar algebraic properties, the most appropriate transition program may be based on the Poisson bracket. Second, in an arbitrary orthogonal coordinate system, the canonical momentum operator is not equal to the corresponding linear momentum operator. As the linear momentum is much more appropriate than the canonical one, all classical quantities containing momentum must be expressed in terms of the linear momentum instead of the canonical one before making the operator substitution. Third, from the results that equations (32a) and (32b) correspond to equations (45a) and (45b), equations (69a) and (69b) to equations (81a) and (81b), and equations (70a) and (70b) to equations (83a) and (83b), etc., the correspondence principle is valid and both the (x)-condition and the symmetrization procedure can be removed.

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